$\sigma^2 = 1.$

Example 6.3.4 *Suppose $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, 1)$. To test $H_0 : \mu = \mu_0$ vs $H_a : \mu = \mu_1$, where $\underline{\mu_1 > \mu_0}$, let us use the Neyman-Pearson Lemma to find the most powerful critical region of size α.*

$$L_0(\mu_0; \underset{\sim}{x}) = f(x_1, \ldots, x_n; \mu_0, 1) = \prod_{i=1}^{n} f(x_i; \mu_0, 1) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{\frac{-(x_i-\mu_0)^2}{2}}$$
$$= (2\pi)^{-\frac{n}{2}} e^{-\sum_{i=1}^{n}(x_i-\mu_0)^2/2}$$

$$L_1(\mu_1; \underset{\sim}{x}) = f(x_1, \ldots, x_n; \mu_1, 1) = \prod_{i=1}^{n} f(x_i; \mu_1, 1) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{\frac{-(x_i-\mu_1)^2}{2}}$$
$$= (2\pi)^{-\frac{n}{2}} e^{-\sum_{i=1}^{n}(x_i-\mu_1)^2/2}$$

The likelihood ratio is:

$$\frac{L_0}{L_1} = \frac{\cancel{(2\pi)^{-\frac{n}{2}}}\, e^{-\sum_{i=1}^{n}(x_i-\mu_0)^2/2}}{\cancel{(2\pi)^{-\frac{n}{2}}}\, e^{-\sum_{i=1}^{n}(x_i-\mu_1)^2/2}} = e^{-\frac{1}{2}\left[\sum_{i=1}^{n}(x_i-\mu_0)^2 - \sum_{i=1}^{n}(x_i-\mu_1)^2\right]}$$

$$\sum_{i=1}^{n} x_i = n\bar{x}$$

$$= e^{-\frac{1}{2}\left[\cancel{\sum_{i=1}^{n} x_i^2} - 2\mu_0 \sum_{i=1}^{n} x_i + n\mu_0^2 - \cancel{\sum_{i=1}^{n} x_i^2} + 2\mu_1 \sum_{i=1}^{n} x_i - n\mu_1^2\right]}$$

$$= e^{n\bar{x}(\mu_0-\mu_1) - \frac{n}{2}(\mu_0^2-\mu_1^2)}$$

Now, we want to find a constant K and a region C such that

$$\frac{L_0}{L_1} = e^{n\bar{x}(\mu_0-\mu_1) - \frac{n}{2}(\mu_0^2-\mu_1^2)} \le K, \quad \text{s.t } (x_1, \ldots, x_n) \in C.$$

$$\Rightarrow n\bar{x}(\mu_0-\mu_1) - \frac{n}{2}(\mu_0^2-\mu_1^2) \le \ln K.$$

$$n\bar{x}(\underbrace{\mu_0-\mu_1}_{\le 0}) \le \ln K + \frac{n}{2}(\mu_0^2-\mu_1^2)$$

$$\bar{x} \ge \frac{\ln K}{n(\mu_0-\mu_1)} + \frac{\frac{n}{2}(\mu_0^2-\mu_1^2)}{n(\mu_0-\mu_1)}$$

$$\bar{x} \ge K^*$$

$$\Rightarrow \begin{cases} \bar{x} \ge K^*, & \text{s.t } (x_1, \ldots, x_n) \in C \\ \bar{x} < K^*, & \text{s.t } (x_1, \ldots, x_n) \in \bar{C}. \end{cases}$$

$X_1, \ldots, X_n \Rightarrow C$

$\downarrow$

$\frac{L_0}{L_1} \le K$

$\downarrow$

$\bar{X} > K^*$

$\downarrow$

$Z > Z_{1-\alpha}$

In fact, we don't really care what the value of K is, we only care for what value of K^*:

$$\overline{x} \geq K^*, \quad (x_1, \ldots, x_n) \in C,$$
$$\overline{x} \leq K^*, \quad (x_1, \ldots, x_n) \notin C.$$

We determine the value of K^* based on the size of test α and the distribution of $\overline{X}$. $\bar{X} \sim N(\mu, \frac{1}{n})$

Given α, determine the K^*

under the $H_0: \mu = \mu_0$. $\bar{X} \sim N(\mu_0, \frac{1}{n})$

$\alpha = P(\text{reject } H_0 \mid H_0 \text{ is true})$

$\alpha = P((x_1, \ldots, x_n) \in C \mid \mu = \mu_0)$

$\alpha = P(\bar{X} > K^* \mid \mu = \mu_0)$

$$\alpha = P\Big(\underbrace{\frac{\bar{X} - \mu_0}{\sqrt{1/n}}}_{\sim N(0,1)} \geq \underbrace{\frac{K^* - \mu_0}{\sqrt{1/n}}}_{= Z_{1-\alpha}} \Big)$$

$K^* = \mu_0 + Z_{1-\alpha} \sqrt{\frac{1}{n}}$

$\Rightarrow$ The most powerful critical region for testing

$H_0: \mu = \mu_0$ vs $H_a: \mu = \mu_1$ $(\mu_1 > \mu_0)$

with $\sigma^2 = 1$,

is given by $C = \{(x_1, \ldots, x_n): \bar{x} \geq \mu_0 + Z_{1-\alpha} \sqrt{\frac{1}{n}}\}$.

Let's consider another application of N-P lemma:

Suppose $X_1, \ldots, X_n \sim$ iid $N(\mu, 1)$. Find the most powerful test for $H_0 : \mu = \mu_0$ vs $H_a : \mu = \mu_1$ ($\mu_1 > \mu_0$) at the α level of significance.

By the N-P lemma, the most powerful test is given by

$$\phi(x_1, \ldots, \psi_n) = \begin{cases} 1 & \text{if } \Lambda \leq k, \\ 0 & \text{if } \Lambda < k, \end{cases}$$

where $\Lambda = \frac{L_0}{L_1} = e^{n\bar{x}\underbrace{(\mu_0 - \mu_1)}_{\leq 0} - \frac{n}{2}(\mu_0^2 - \mu_1)}$

where

$$\Lambda \leq K$$

$$\Leftrightarrow \quad \bar{x} \geq K^*$$

large $\bar{x}$, large μ $\Rightarrow$ more favor to Ha

The value of k is determined by the size of test α. so as well as K^*

$$\alpha = P(\Lambda \leq K \mid H_0 \text{ is true})$$

$$= P(\bar{x} > K^* \mid \mu = \mu_0)$$

Under $H_0 : \mu = \mu_0$,

pivot statistic $\longrightarrow$ $Z = \frac{\bar{X} - \mu_0}{\sqrt{1/n}} \sim N(0,1)$

$$\alpha = P\left(\frac{\bar{x} - \mu_0}{\sqrt{1/n}} \geq \frac{K^* - \mu_0}{\sqrt{1/n}} \right)$$

$$= P(Z \geq K^{**}).$$

where $K^{**} = Z_{1-\alpha}$.

Thus, the rejection region $\{\Lambda \leq k\}$ is equivalent to

$$\{(x_1, \cdots, x_n) : Z \geq Z_{1-\alpha}\}. \qquad Z = \frac{\bar{x} - \mu_0}{\sqrt{1/n}}$$

The most powerful test at α-level is given by

$$\phi(X_1, \cdots, X_n) = \begin{cases} 1 & \text{if } Z \geq Z_{1-\alpha} \\ 0 & \text{o.w} \end{cases}$$

6.4 The power function of a test

In general, type I error is more series than the type II error. Therefore, we control the α at a pre-specified level, then find a critical region, C, based on the given α that maximizes the power. By doing so, the probability of type I error is controlled at α level and the power $= (1-\beta)$ is maximized.

The Neyman-Pearson lemma is for testing a simple null hypothesis $H_0 : \theta = \theta_0$ against a simple alternative hypothesis $H_a : \theta = \theta_1$. We might want to test, say, $H_0 : \theta < \theta_0$ a composite null hypothesis against $H_a : \theta > \theta_0$, a composite alternative hypothesis, pair of composite hypotheses.

Let us consider a framework:

$$H_0: \theta \in \Theta_0 \quad vs \quad H_a: \theta \in \Theta_1$$

$$\text{where} \quad \Theta_0 \cap \Theta_1 = \emptyset$$

Definition 6.4.1 *The **power function**, denoted as $\pi(\theta)$, of a test of $H_0 : \theta \in \Theta_0$ against $H_a : \theta \in \Theta_1$ is given by*

$$\pi(\theta) = \begin{cases} \alpha(\theta) & \textit{for value of } \theta \textit{ assumed under } H_0, \\ 1 - \beta(\theta) & \textit{for value of } \theta \textit{ assumed under } H_a. \end{cases}$$

The power function, $\pi(\theta)$, is in fact, the probability of rejecting the H_0 for a given value of θ:

$$\pi(\theta) = P(\textit{reject } H_0|\theta)$$

Example 6.4.2 *Suppose $X \sim bin(5, \theta)$. We want to test*

$$H_0 : \theta \leq \frac{1}{2} \qquad vs \qquad H_a : \theta > \frac{1}{2}$$

Partitioning of ~~sample~~ parameter space:

$$\Theta = \{\theta : 0 < \theta < 1\}$$

$$\Theta_0 = \{\theta : 0 < \theta \leq \tfrac{1}{2}\}$$

$$\Theta_1 = \{\theta : \tfrac{1}{2} < \theta < 1\}$$

Suppose our critical region is $C_1 = \{x : x \in \{4, 5\}\}$. Then the power function is given by:

$X \sim Bin(5, \theta)$

$f(x; \theta) = \binom{5}{x} \theta^x (1-\theta)^{5-x}$, $x = 0, 1, 2, 3, 4, 5$

power functn: $\pi_1(\theta) = P(\text{reject } H_0 \mid \theta)$

$= P(X = 4 \text{ or } 5 \mid \theta)$

$= P(X=4 \mid \theta) + P(X=5 \mid \theta)$

$= \binom{5}{4} \theta^4 (1-\theta) + \binom{5}{5} \theta^5 (1-\theta)^0$

$= 5\theta^4 (1-\theta) + \theta^5$

$= 5\theta^4 - 4\theta^5$, $\quad 0 < \theta < 1$

We sketch the power function for the given critical region $C_1 = \{x : x \in \{4, 5\}\}$.

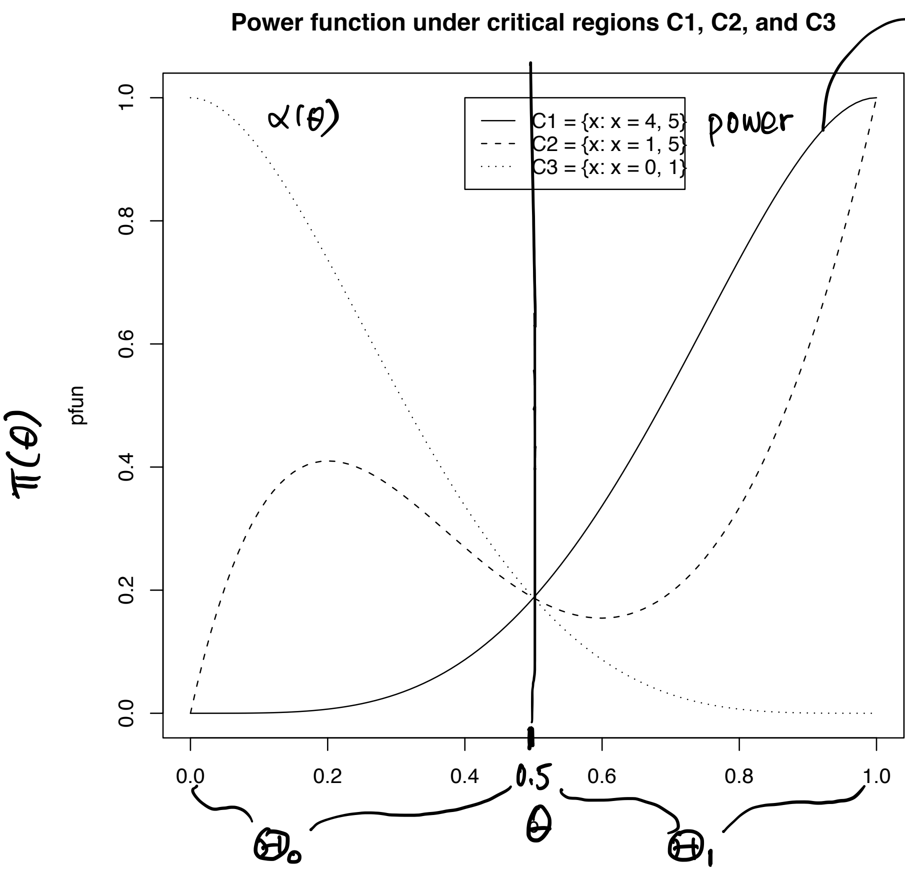
	θ	$\pi_1(\theta) = 5\theta^4 - 4\theta^5$
$\theta \in \Theta_0$	0.1	$\alpha(0.1) = 5(0.1)^4 - 4(0.1)^5 = 0.00045$
	0.2	$\alpha(0.2) = 5(0.2)^4 - 4(0.2)^5 = 0.00675$
	...	...
	0.5	$\alpha(0.5) = 5(0.5)^4 - 4(0.5)^5 = 0.1875$
$\theta \in \Theta_1$	0.6	$1 - \beta(0.6) = 5(0.6)^4 - 4(0.6)^5 =$ 0. [illegible] .69
	...	...
	0.9	$1 - \beta(0.9) = 5(0.9)^4 - 4(0.9)^5 =$ [illegible] 11854

We sketch the power function for other critical regions $C_2 = \{x : x \in \{1, 5\}\}$ and $C_3 = \{x : x \in \{0, 1\}\}$.

$\pi_2(\theta) = P(X = 1, 5 \mid \theta)$

$\pi_3(\theta) = P(X = 0, 1 \mid \theta)$

$H_0: \theta \leq \frac{1}{2}$ vs $H_a: \theta > \frac{1}{2}$.



C_1 is the best critical region. C_1 gives the smallest α and the highest power

Definition 6.4.3 *Given a pre-specified significance level α, if a test*

$$\phi(x_1, \ldots, \psi_n) = \begin{cases} 1 & \text{if } (x_1, \ldots, x_n) \in C, \\ 0 & \text{if } (x_1, \ldots, x_n) \notin C, \end{cases}$$

type 1 error rate

satisfies $P($reject $H_0 | H_0$ is true$) \leq \alpha$, then the test is called an α ***level significant test****.*

Definition 6.4.4 *An α level significant test with the smallest β (or the greatest power) is called the* ***uniformly most powerful test*** *(UMPT).*

1). pre-specified α: size of test $\xrightarrow{\text{work out}}$ C, the critical region.

2) prespecified C the critical region $\xrightarrow{\text{work out}}$ α, β, power, power function.

Remarks:

1. There could be multiple tests (rejection regions) at a given α level; we want the one that maximizes the power.

2. Unfortunately, uniformly most powerful tests rarely exist when testing a simple null hypothesis versus a composite alternative hypothesis, e.g.,

   $H_0: \mu = \mu_0 \quad vs \quad H_a: \mu > \mu_0.$

3. When testing a simple null hypothesis versus a simple alternative hypothesis, e.g.,

   $H_0: \mu = \mu_0 \quad vs \quad H_a: \mu = \mu_1 \quad (\mu_1 > \mu_0)$

   the N-P lemma gives the uniformly most powerful test.

   N-P lemma can only be applicable when comparing $\mu = \mu_0$ and $\mu = \mu_1$.

6.5 The likelihood ratio tests

The Neyman-Pearson lemma provides a method for constructing the most powerful critical region for testing:

$H_0: \theta = \theta_0 \quad vs \quad H_a: \theta = \theta_1 \quad (\theta_0 \neq \theta_1)$ use $\frac{L_0}{L_1}$

We now present a general method, the likelihood ratio test (LRT), for constructing critical regions for the hypothesis tests that consist of composite hypothesis such as:

$H_0: \theta \in \Theta_0 \quad vs \quad H_a: \theta \in \Theta_1$

$\Theta_0 \cap \Theta_1 = \phi$, can we use $\frac{L_0}{L_1}$? eg $H_0: \theta = \theta_0$ vs $H_a: \theta \neq \theta_0$

LRTs are generalization of the Neyman-Pearson lemma, but they are not necessarily uniformly most powerful. LRTs compare the maximum likelihood under H_0 with the unrestricted maximum likelihood for all values in the parameter space, that is $\theta \in \Theta$.

$\max L_0 = \max_{\theta \in \Theta_0} L(\theta) \leftarrow$ restricted Maximum Likelihood

$\max L = \max_{\theta \in \Theta} L(\theta) \leftarrow$ unrestricted maximum Likelihood

$\Theta = \Theta_0 \cup \Theta_1$

Suppose we have a random sample $(X_1, \ldots, X_n) \overset{iid}{\sim} f(x;\theta)$. The maximum likelihood under H_0 is given by

$$\max L_0 = \prod_{i=1}^{n} f(x_i; \tilde{\theta})$$

where $\tilde{\theta} \in \Theta_0$, is the MLE of θ within Θ_0.

($\tilde{\theta}$: restricted MLE of θ)

The maximum likelihood for all values of $\theta \in \Theta$, is given by

$$\max L = \prod_{i=1}^{n} f(x_i; \hat{\theta})$$

where $\hat{\theta}$ is the MLE of θ, $\theta \in \Theta$

($\hat{\theta}$: unrestricted MLE of θ).

Then, their ratio

$$\Lambda = \frac{\max L_0}{\max L}$$

is referred to the **likelihood ratio statistic**.

Suppose we have

$$\underset{\text{local maximum}}{\max L_0 = L(\tilde{\theta})} \quad \leq \quad \underset{\text{global maximum}}{\max L = L(\hat{\theta})}.$$

where $\tilde{\theta}$ is restricted MLE

and $\hat{\theta}$ is unrestricted MLE.

The equality holds iff $\tilde{\theta} = \hat{\theta}$.

- There are two scenarios to consider:
  - If H_0 is true, we expect:

$H_0: \theta \in \Theta_0$ $\quad \dfrac{\max L_0}{\max L} = 1$, $\quad \max L_0 \approx \max L$

– If H_0 is false, we expect:
$H_a: \theta \in \Theta,$ $\max L_0 < \max L$ $0 < \frac{\max L_0}{\max L} < 1$

- The ratio
$$\Lambda = \frac{\max L_0}{\max L}$$
is bounded between 0 and 1.

- If $\Lambda \approx 0$, we would like to reject H_0;
if $\Lambda \approx 1$, we would like to accept H_0.

Definition 6.5.1 *If $\Theta = \Theta_0 \cup \Theta_1$ and $\Theta_0 \cap \Theta_1 = \emptyset$, and if*
$$\Lambda = \frac{\max L_0}{\max L} = \frac{L(\tilde{\theta})}{L(\hat{\theta})},$$
then the critical region
$$\Lambda \leq k$$
*where $0 < k < 1$, is a **likelihood ratio test** for testing $H_0 : \theta \in \Theta_0$ against $H_a : \theta \in \Theta_1$.*

Example 6.5.2 *Suppose we have a random sample $(X_1, \ldots, X_n)$ from a $N(\mu, \sigma^2)$. Find the critical region of the likelihood ratio test for testing*
$$H_0 : \mu = \mu_0 \qquad vs \qquad H_a : \mu \neq \mu_0.$$
simple hypothesis composite hypothesis.

Since the only choice for μ under H_0 is μ_0, we have
$\max L_0 = \prod_{i=1}^{n} f(x_i; \tilde{\mu}, \sigma^2) = \prod_{i=1}^{n} f(x_i; \mu_0, \sigma^2) = L(\mu_0, \sigma^2)$
$= (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\sum_{i=1}^{n}(x_i - \mu_0)^2 / 2\sigma^2}$

$\tilde{\mu}$, the restricted MLE of μ is μ_0, because under H_0: $\mu = \mu_0$

Further, we know the MLE of μ is $\hat{\mu} = \overline{X}$, so have
$\max L = \prod_{i=1}^{n} f(x_i; \hat{\mu}, \sigma^2) = \prod_{i=1}^{n} f(x_i; \overline{x}, \sigma^2) = L(\overline{x}, \sigma^2)$
$= (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\sum_{i=1}^{n}(x_i - \overline{x})^2 / 2\sigma^2}$

The likelihood ratio statistic becomes

$$\Lambda = \frac{\max L_0}{\max L} = \frac{\cancel{(2\pi\sigma^2)^{-\frac{n}{2}}}\, e^{-\sum_{i=1}^{n}(x_i-\mu_0)^2/2\sigma^2}}{\cancel{(2\pi\sigma^2)^{-\frac{n}{2}}}\, e^{-\sum_{i=1}^{n}(x_i-\bar{x})^2/2\sigma^2}}$$

$$= e^{-\frac{1}{2\sigma^2}\left\{\sum_{i=1}^{n}(x_i-\mu_0)^2 - \sum_{i=1}^{n}(x_i-\bar{x})^2\right\}}$$

$$= e^{-\frac{1}{2\sigma^2}\left\{\cancel{\sum_{i=1}^{n} x_i^2} - 2n\bar{x}\mu_0 + n\mu_0^2 - \cancel{\sum_{i=1}^{n} x_i^2} + \cancel{2}n\bar{x}^2 - \cancel{n\bar{x}^2}\right\}}$$

$$= e^{-\frac{n}{2\sigma^2}(\bar{x}-\mu_0)^2}$$

Hence the critical region of the likelihood ratio test can be derived as

$$\Lambda = e^{-\frac{n}{2\sigma^2}(\bar{x}-\mu_0)^2} \le k, \quad \text{s.t. } (x_1,\dots,x_n)\in C$$

$$\Rightarrow \quad (\bar{x}-\mu_0)^2 \ge \frac{-2\sigma^2}{n}\ln k \qquad 0<k<1$$

$$\Rightarrow \begin{cases} \bar{x}-\mu_0 \ge \sqrt{\frac{-2\sigma^2}{n}\ln k} = k^* \\ \bar{x}-\mu_0 \le -\sqrt{\frac{-2\sigma^2}{n}\ln k} = -k^* \end{cases}$$

$$|\bar{x}-\mu_0| \ge k^* \quad \text{or.} \quad \begin{cases} \bar{x}-\mu_0 \ge k^* \\ \bar{x}-\mu_0 \le -k^* \end{cases}$$

We determine the critical region by the size of the test, α:

$$\alpha \xrightarrow{\text{work out}} C.$$

$$\alpha = P(|\bar{x}-\mu_0| \ge k^* \mid \mu=\mu_0)$$

We know $\overline{X} \sim N(\mu_0, \frac{\sigma^2}{n})$ under H_0, so we have:

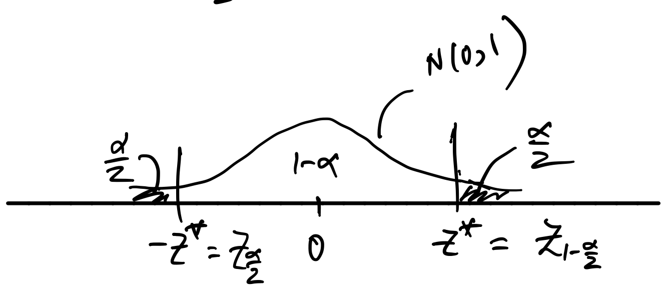
$$\alpha = P\left(\frac{|\overline{X} - \mu_0|}{\sigma/\sqrt{n}} \geq \frac{k^*}{\sigma/\sqrt{n}} \right) \qquad Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$\alpha = P\left(|Z| \geq \frac{k^*}{\sigma/\sqrt{n}} \right) \qquad Z^* = \frac{k^*}{\sigma/\sqrt{n}}.$$

$$\alpha = P(Z \geq Z^* \text{ or } Z \leq -Z^*)$$

$$\alpha = P(Z \geq Z^*) + P(Z \leq -Z^*)$$

$$Z^* = Z_{1-\frac{\alpha}{2}} \quad , \qquad -Z^* = Z_{\frac{\alpha}{2}}$$



$$\Lambda \leq k$$
$$\Downarrow$$
$$|\overline{X} - \mu_0| \geq k^*$$
$$\Downarrow$$
$$|Z| = \left|\frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}\right| \geq Z^*$$

Then, the critical region for LRT is

$$C = \left\{ (x_1, \dots, x_n) : |Z| = \frac{|\overline{X} - \mu_0|}{\sigma/\sqrt{n}} \geq Z_{1-\frac{\alpha}{2}} \right\}$$

In Example 6.5.2, when the random sample is from a normal distribution, it is relatively easy to find the critical region for the test, since we know the distribution of the pivotal quantity for estimating the parameter. This means we don't have to derive the distribution of Λ. However, the distribution of Λ is often difficult to derive, and thus, it is often difficult to determine the critical value k. In this case, we can use the following approximation.

$X_1, \cdots, X_2 \overset{iid}{\sim} f(x;\theta)$ $\quad$ $f(x;\theta)$ is not normal.

Theorem 6.5.3 *For a large sample size, n,*

$$-2\ln\Lambda = -2\ln\left(\frac{\max L_0}{\max L}\right) \overset{approx}{\sim} \chi^2_1.$$

With reference to Example 6.5.2, we can find critical region using Theorem 6.5.3.

$X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$H_0: \mu = \mu_0 \quad$ vs $\quad H_a: \mu \neq \mu_0.$

$\Lambda = e^{-\frac{n}{2\sigma^2}(\bar{x}-\mu_0)^2}$

$$\boxed{\begin{aligned} -2\ln\Lambda &= +\frac{n}{\cancel{2}\sigma^2}(\bar{x}-\mu_0)^2 \times (+\cancel{2}) = \frac{n(\bar{x}-\mu_0)^2}{\sigma^2} \\ &= \frac{(\bar{x}-\mu_0)^2}{\sigma^2/n} \sim \chi^2_1 \end{aligned}}$$

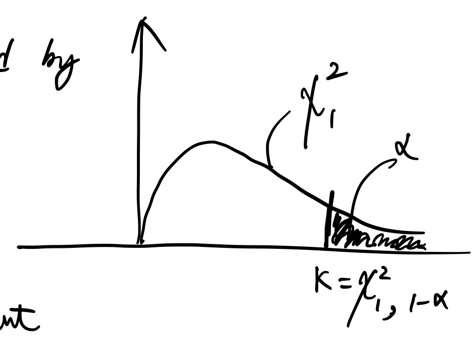
$\bar{X} \sim N(\mu_0, \frac{\sigma^2}{n})$

$Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$

$Z^2 = \frac{(\bar{X}-\mu_0)^2}{\sigma^2/n} \sim \chi^2_1$

Then the critical region can be determined by

$$\alpha = P(\underbrace{-2\ln\Lambda}_{\sim\,\chi^2_1} \geq \underbrace{k}_{=\,\chi^2_{1,1-\alpha}})$$



$C = \{(x_1, \cdots, x_n): \frac{(\bar{x}-\mu_0)^2}{\sigma^2/n} \geq \chi^2_{1,1-\alpha}\}$ $\overset{\text{equivalent}}{\Longleftrightarrow}$ C on Pg 110

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