STAT*3110 - WINTER2020

INTRODUCTORY MATHEMATICAL STATISTICS II

Zeny Feng

Department of Mathematics and Statistics,
University of Guelph

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Contents

Chapter 1

Review of Probability and Distribution (STAT 3100)

1.1 Probability

1.1.1 Probability - frequentist approach

1. Random experiment

   - all possible outcomes can be listed.

   - the outcome is generally uncertain.

2. Sample Space, S, a set of all possible outcomes.

3. Random event, A

   - a possible outcome
   - a subset of sample space

   - event space, $\mathcal{A}$, a collection of all possible events.

4. Probability measure, $P(\cdot)$, a function defined over the sample space mapping to [0,1]

1.1.2 Frequency interpretation

Example: $\mathrm{P}(A) = 0.9$.

1.1.3 Independence of two events

A and B are independent

1.2 Random variables and their distributions

Definition 1.2.1 *A random variable is a function that maps from sample space to the real line.*

Probability distribution function of a random variable X

1. Discrete random variable X:

   - Probability mass function, pmf: $f(x) = \mathrm{P}(X = x)$

   - $\mathrm{P}(X \in A) = \sum_{x \in A} \mathrm{P}(X = x)$

2. Continuous random variable X

   - Probability density function, pdf:

- Note that, $\mathrm{P}(X = x) = 0$.
- In a general form,

$$\mathrm{P}(X \in A) = \int_A f(x)dx$$

1.2.1 Cumulative distribution function (cdf).

$$F(x) = \mathrm{P}(X \leq x), \qquad 0 \leq F(x) \leq 1.$$

$F(x)$ is a non decreasing function:

1. Discrete case: $F(X)$ is a step function.

2. Continuous case:

1.3 Expectation, variance and moments

1.3.1 Expectation: mean, average.

1. Expectation of the function of X.

2. Expectation of X.

3. Expectation of the linear combination.

1.3.2 Moments

1. rth moment.

2. rth central moment.

3. Chebyshev's inequality:
   If $\mathrm{E}(X) = \mu$ and $\mathrm{var}(X) = \sigma^2$, then, $\forall \varepsilon > 0$,

$$\mathrm{P}(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

4. Markov inequality: $\forall \varepsilon > 0$,

$$\mathrm{P}(X \geq \varepsilon) \leq \frac{\mathrm{E}(X)}{\varepsilon}.$$

5. If X and Y are two random variables

$$\text{var}(aX + bY) = a^2\text{var}(X) + 2ab\text{cov}(X, Y) + b^2\text{var}(Y)$$

1.4 Moment generating function (mgf)

Definition 1.4.1 *The* ***moment generating function*** *of a random variable X, $M_X(t)$ is given by*

$$M_X(t) = E(e^{tX}), \qquad \textit{for } t \in (-\delta, \delta)$$

where δ is a fixed value.

Example 1.4.2 *Find the mgf for $X \sim Bin(n, p)$.*

Example 1.4.3 *Find the mgf for $X \sim Exponential(\lambda)$.*

1. Note that, the mgf does not always exists. The characteristic function $\varphi_X(t) = \mathrm{E}(e^{itX}), i = \sqrt{-1}$, always exists for any random variable.

2. Properties of mgf.

   - $Y = aX + b$, then,

   $$M_Y(t) = e^{bt} M_X(at)$$

   Example: $Z \sim N(0,1), M_Z(t) = e^{t^2/2}$. If $X \sim N(\mu, \sigma^2)$, then

   - X and Y are independent, then

   $$M_{X+Y}(t) = M_X(t) M_Y(t).$$

   - Use mgf to find the rth moment, $E(X^r)$.

   - Uniqueness of mgf.
   If $M_X(t) = M_Y(t)$, then $f_X(x) = f_Y(y), \forall x = y$.

1.5 Common distribution

1.5.1 Discrete random variables

1. Bernoulli (p) and binomial (N, p) distributions

2. Geometric and negative binomial distributions

3. Hypergeometric distribution

- Without replacement

- With replacement

4. Poisson distribution
   A limiting form of binomial distribution when $n \to \infty, p \to 0$, while $\lambda = np$ is constant, the number of successes $X \sim \text{Poisson}(\lambda)$.

1.5.2 Continuous random variables

1. Uniform$[a, b]$, $f(x) = \frac{1}{b-a}$, $a \leq x \leq b$.

2. $X \sim$ Exponential(θ).

   Memoryless property: $\mathrm{P}(X > t + s | X > s) = \mathrm{P}(X > t)$

3. Normal distribution, $X \sim N(\mu, \sigma^2)$.

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, X \in (-\infty, \infty).$$

4. $X \sim \text{Gamma}(\alpha, \beta), \alpha, \beta > 0,$

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x > 0$$

$$\text{E}(X) = \alpha\beta, \qquad \text{var}(X) = \alpha\beta^2$$

- Special case of a Gamma distribution

$$X \sim \text{Exponential}(\beta) = \text{Gamma}(1, \beta)$$

$$X \sim \chi^2_d = \text{Gamma}(\frac{d}{2}, 2), \qquad d = 1, 2, 3, ...$$

5. $X \sim \text{Beta}(\alpha, \beta), \alpha, \beta > 0,$

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x, 1$$

$$\text{E}(X) = \frac{\alpha}{\alpha+\beta}, \qquad \text{var}(X) = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}.$$